THE CONTINUUM LIMIT OF THE NON-COMMUTATIVE $\lambda\phi^4$ MODEL

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We present a numerical study of the $\lambda\phi^4$ model in three Euclidean dimensions, where the two spatial coordinates are non-commutative (NC). We first show the explicit phase diagram of this model on a lattice. The ordered regime splits into a phase of uniform order and a "striped phase". Then we discuss the dispersion relation, which allows us to introduce a dimensionful lattice spacing. Thus we can study a double scaling limit to zero lattice spacing and infinite volume, which keeps the non-commutativity parameter constant. The dispersion relation in the disordered phase stabilizes in this limit, which represents a non-perturbative renormalization. From its shape we infer that the striped phase persists in the continuum, and we observe UV/IR mixing as a non-perturbative effect.

1 The Non-Commutative Plane

We consider a NC plane given by Hermitian coordinate operators $\hat{x}_\mu$, which obey

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta\epsilon_{\mu\nu} \; . \tag{1}$$

We impose a (fuzzy) lattice structure on this plane by means of the operator identity

$$\exp\left(i\frac{2\pi}{a}\hat{x}_\mu\right) = \hat{1\!\!1} \; . \tag{2}$$

Now the periodicity of the momentum components k_μ implies

$$\theta k_\mu / 2a \in \mathbb{Z} \; . \tag{3}$$

Hence the lattice is automatically periodic, say over the lattice volume $N \times N$. Then the non-commutativity parameter corresponds to

$$\theta = N a^2 / \pi \; . \tag{4}$$

We see that the limits to the continuum ($a \to 0$) and to the infinite volume ($Na \to \infty$) should be taken simultaneously, if we want to keep θ finite. In particular, we are interested in the double scaling limit $a \to 0$, $N \to \infty$, which keeps $\theta = const.$

2 The 3d NC $\lambda\phi^4$ Model

In the star product formulation, the action of the NC $\lambda\phi^4$ model in Euclidean space takes the form

$$S[\phi] = \int d^dx \left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\right] .$$

Since the star product does not affect the bilinear terms, λ determines the strength of NC effects. We consider this model in $d = 3$ with a commutative Euclidean time and a NC plane. Its formulation on a N^3 lattice can be mapped onto a matrix model[1] of the form

$$S[\bar{\phi}] = N\mathrm{Tr}\sum_{t=1}^{N}\left\{\frac{1}{2}\sum_{\mu=1}^{2}\left[\Gamma_\mu\bar{\phi}(t)\Gamma_\mu^\dagger - \bar{\phi}(t)\right]^2 + \frac{1}{2}\left[\bar{\phi}(t+1) - \bar{\phi}(t)\right]^2 + \frac{m^2}{2}\bar{\phi}^2(t) + \frac{\lambda}{4}\bar{\phi}^4(t)\right\} , \tag{5}$$

where each time site $t = 1 \dots N$ accommodates a Hermitian $N \times N$ matrix $\bar{\phi}(t)$. The "twist eaters" Γ_μ provide a shift by one lattice unit in a spatial direction, if they obey the 't Hooft-Weyl algebra

$$\Gamma_\mu\Gamma_\nu = Z_{\nu\mu}\Gamma_\nu\Gamma_\mu \; . \tag{6}$$

The twist $Z_{\mu\nu} = Z_{\nu\mu}^*$ is a phase factor; in our formulation it reads $Z_{12} = \exp(i\pi(N+1)/N)$.

3 The Phase Diagram

In contrast to the star product formulation, the matrix formulation of eq. (5) is suitable for Monte Carlo simulations. Our numerical results are described in Ref. [2], and in several proceeding contributions as well as a Ph.D. thesis.[3] Similar techniques were applied to arrive at non-perturbative results for 2d NC field theories, in particular for the $\lambda\phi^4$ model on a NC plane[4,2] and on a fuzzy sphere,[5] and for NC QED_2.[6]

In the 3d model described above, we first explored the phase diagram. We found it to be stable

for $N \gtrsim 25$ in the plane spanned by the axes $N^2 m^2$ and $N^2\lambda$, see Figure 1.

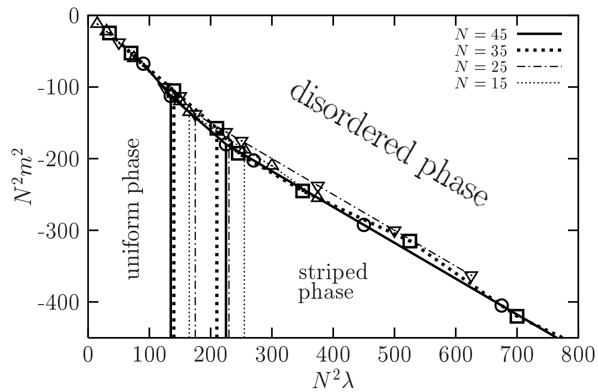

Figure 1: *The phase diagram of the 3d NC $\lambda\phi^4$ model, obtained from numerical simulations on a N^3 lattice.*

Strongly negative m^2 leads to ordering. At weak λ the order is uniform as in the commutative world, whereas at stronger λ (corresponding to an amplified θ) stripe patterns dominate. This picture agrees with analytic conjectures.[7] The occurrence of a striped phase is a qualitative difference from the commutative $\lambda\phi^4$ model, though similar effects are known for instance in the (commutative) Gross-Neveu model at large chemical potential[8] and in ferromagnetic superconductors.

Our results for the hysteresis suggest that the uniform-striped transition is of first order, while both disorder-order transitions are of second order. The formation of stripes was observed by introducing a momentum dependent order parameter. The problem in this direct consideration is that at our values of N usually just two stripes can be observed as manifestly stable. However, multi-stripe patterns are supposed to dominate the striped phase at large volumes. We will demonstrate this behavior in an indirect way in Section 5.

4 Dispersion Relation

The correlation functions with a spatial separation have a fast but non-exponential decay;[2] apparently it is distorted by the NC geometry. In Euclidean time direction, however, the decay of the correlator turned out to be exponential. At fixed spatial momenta $\vec{p} = (p_1, p_2)$ this allows us to determine the energy E and thus the dispersion relation $E^2(\vec{p}^{\,2})$. It is most instructive to look at it in the disordered phase — which does not suffer much from finite size effects — close to the ordering transition. Figure 2 shows examples close to the uniform order (on top) and close to the striped order (below). The former follows the familiar linear shape, whereas the latter has its energy minimum at non-zero momentum. Clearly for decreasing m^2 the minimal mode condenses, giving rise to a stripe pattern.

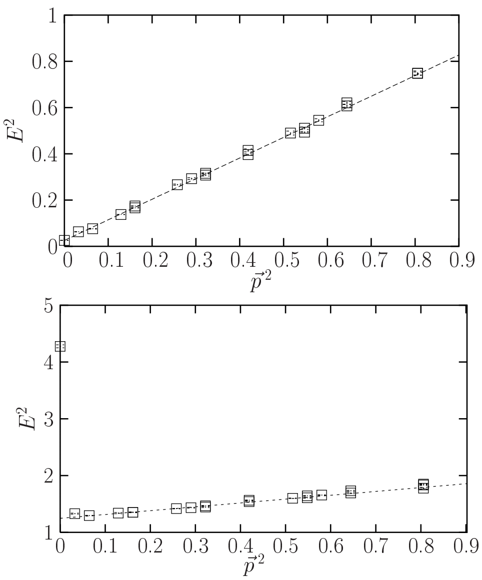

Figure 2: *The dispersion relation in the disordered phase, close to ordering, at $N = 35$. For $\lambda = 0.06$ (on top) we obtain the usual linear dispersion, but for $\lambda = 100$ (below) the energy minimum moves to a finite momentum.*

5 Continuum limit

In order to obtain a "physical scale" we have to identify a dimensionful quantity. To this end we consider the planar limit $N \to \infty$ at fixed λ and m^2. This pushes the IR jump in the dispersion towards zero, and we obtain (for all finite momenta) a linear dispersion of the form $E^2 = M_{\rm eff}^2 + \vec{p}^{\,2}$. Measuring now $M_{\rm eff}^2$ at fixed λ but varying m^2 we observed a linear dependence,

$$M_{\rm eff}^2|_{\lambda=const.} = \mu^2 + \gamma m^2 \ , \tag{7}$$

which corresponds to the critical exponent $\nu = 1/2$. For instance, at $\lambda = 50$ we found the critical parameter $m_c^2 = -\mu^2/\gamma = -15.01(8)$.

The continuum limit is now taken such that the dimensionful effective mass, $M_{\rm eff}/a$, remains constant (a being the lattice spacing). Hence the double scaling limit means $N \to \infty$ and $m^2 \to m_c^2$ such that

$N(m^2 - m_c^2) = const$. In our study we chose the latter constant as 100, which implies $\theta = 100\gamma/\pi = 9.77(6)(a/M_{\rm eff})^2$.

It turns out that the dispersion relation stabilizes in this double limit at all finite momenta, which demonstrates the non-perturbative renormalizability of this model. Figure 3 (on top) shows the energy minimum around $\vec{p}^{\,2}/a^2 \lesssim 0.1$. This translates into a dominant, finite stripe width, so after condensation we expect an infinite number of stripes in the double scaling limit to the continuum and infinite volume.

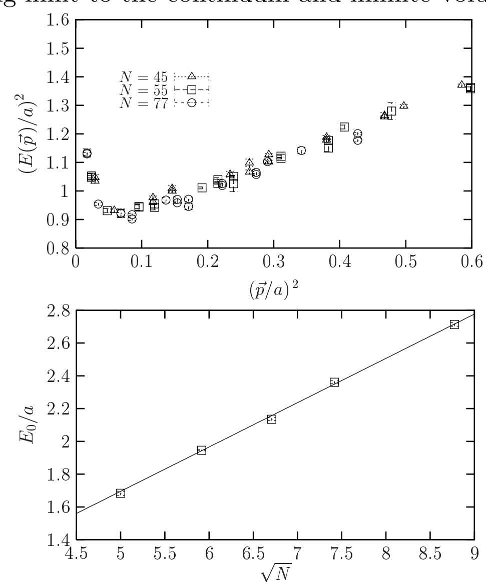

Figure 3: *On top: The dispersion relation as we approach the double scaling limit. Its stabilization indicates non-perturbative renormalizability. The dip at finite* $(\vec{p}/a)^2$ *shows that the striped phase survives this limit. Below:* E_0/a *diverges linearly in* $\sqrt{N} \propto 1/a$*, in agreement with UV/IR mixing.*

In this limit, the rest energy $E_0/a = E(\vec{p} = \vec{0})/a$ diverges linearly in $\sqrt{N} \propto 1/a$, see Figure 3 (below). Also the UV divergence is linear in this model, hence our observation agrees perfectly with the concept of UV/IR mixing, which is known from perturbation theory.[9] Here this mixing is observed as a non-perturbative effect, so it belongs to the very nature of the system.

6 Conclusions

We have shown that the NC 3d $\lambda\phi^4$ model — with two NC spatial direction and a Euclidean time — is *non-perturbatively renormalizable*, and that its phase diagram includes a *striped phase*, which is there to stay in the continuum limit (more precisely: in the double scaling limit to zero lattice spacing and infinite volume, at a fixed non-commutativity parameter θ).

The striped phase implies the spontaneous breaking of translation symmetry. It also exists in the 2d version of this model (omitting the time direction);[4,2] note that NC field theories are non-local, hence the Mermin-Wagner Theorem does not apply. At $\theta \to \infty$ perturbation theory suggests a commutative behavior (in this case the equivalence to a large N matrix field theory). However, in our phase diagram the uniform order does not return at large λ (which corresponds to large θ); in the case of spontaneous symmetry breaking the perturbative argument is not valid, because it does not capture an expansion around the striped ground state.[10]

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